Classification of radial solutions to equations related to Caffarelli-Kohn-Nirenberg inequalities

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Abstract

This article studies the qualitative and quantitative properties of radial solutions to an elliptic equation related to the Euler-Lagrange equations for certain sharp Caffarelli-Kohn-Nirenberg inequalities. Namely, we examine the equation

$$-div(|x|^a Du) = |x|^b u^p,\ u > 0,\ \text{ in } \mathbb{R}^N,$$

where $p > 1$, $N \geq 2$, $N - 2 + a \geq 0$ and $b > -N$. The main results establish the properties of radially symmetric solutions including existence, uniqueness, and classification results as well as results on the asymptotic and intersecting behavior of such solutions.

Keywords: Asymptotic, Caffarelli-Kohn-Nirenberg inequalities; classification of solutions, Hénon-Lane-Emden equation; Liouville theorem; Hardy-Sobolev inequality; phase plane, radial solution.

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1 Introduction

We consider the following nonlinear elliptic equation and its positive solutions,

$$div(|x|^a Du) + |x|^b u^p = 0,\ u > 0,\ \text{ in } \mathbb{R}^N, \tag{1.1}$$

where throughout the paper $N \geq 2$, $p > 1$, $b > -N$ and $N - 2 + a \geq 0$. Thus, $a, b > -N$, which ensure the coefficients are locally integrable. In

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certain cases, equation (1.1) is the Euler-Lagrange equation for the following Caffarelli-Kohn-Nirenberg inequalities established in [5] (see [6, 8, 36] for further background discussion).

Theorem A. *Let $p \geq 1$ $N \geq 3$ and $b, a-2 > -N$. There exists a positive constant $\mathcal{N} = \mathcal{N}(N, a, b)$, depending only on N, a and b, such that for each $u \in C_c^1(\mathbb{R}^N)$,*

$$\mathcal{N}\Big(\int_{\mathbb{R}^N} |x|^b |u|^{p+1}\, dx\Big)^{2/(p+1)} \leq \int_{\mathbb{R}^N} |x|^a |Du|^2\, dx, \tag{1.2}$$

where

$$a - 2 \leq 2b/(p+1) \leq a \quad \text{and} \quad \frac{N+b}{p+1} + 1 = \frac{N+a}{2}.$$

This family of estimates is also called the Hardy-Sobolev inequalities, since the endpoint cases recover the classical Sobolev and Hardy inequalities. Likewise, equation (1.1) becomes the so-called Lane-Emden equation if $a, b = 0$ and the Hardy-Hénon equation if $a = 0$. In addition to being connected with sharp Sobolev and Caffarelli-Kohn-Nirenberg inequalities, the Lane-Emden and Hardy-Hénon equations arise in numerous fundamental problems, e.g., in the study of astrophysical models [23], the Yamabe problem from conformal geometry [1, 26, 35, 37], a priori bounds and blow up analysis for general elliptic problems [19, 30, 31], and the properties of solutions to time-dependent problems [2, 20, 32, 33, 40]. As the literature devoted to the two equations alone is quite large, listing all relevant references is not feasible though some notable references, in addition to the papers mentioned shortly below, are [10, 13, 15, 22, 27, 28, 34, 38, 39].

The aim of this paper is to complement previous existence and nonexistence results for problem (1.1) by establishing a detailed asymptotic analysis of its radially symmetric solutions. In addition to the aforementioned applications, our motivation for obtaining such results for the elliptic problem is motivated by their importance in understanding the finite time blow-up of solutions and the stability of positive steady states to closely related parabolic equations [20, 34, 40].

We now introduce some basic terminology used in this paper. We let Ω denote any open subset of $\mathbb{R}^N$ containing the origin. For $q \geq 1$, the weighted Lebesgue space $L_b^q(\Omega)$ denotes the collection of functions f such that $|x|^{b/q} f \in L^q(\Omega)$ with norm $\|f\|_{L_b^q(\Omega)} = (\int_\Omega |x|^b |f|^q\, dx)^{1/q}$. The space $L_{b,loc}^q(\Omega)$ denotes the collection of functions f that are locally in $L_b^q(\Omega)$, i.e., $f \in L_b^q(K)$ for every compact subset K of Ω. We denote by $H^{1,a}(\Omega)$ the function space given by the completion of $C_c^\infty(\Omega)$ with respect to the norm

$$\|f\| = \|Df\|_{L_a^2(\Omega)}.$$

We let $H_{loc}^{1,a}(\Omega)$ denote the space of functions locally in $H^{1,a}(\Omega)$ in the same sense as with the weighted Lebesgue spaces. By a standard variational formulation [6, 10], the optimizers associated with the sharp constant in (1.2) naturally belong to such weighted function spaces and satisfy (1.1) in some weak sense–a term we define more precisely shortly below. Generally speaking, we focus mainly on locally bounded (radial) solutions satisfying equation (1.1) in the following distributional sense.

Definition. *We say a positive function u is a* **weak solution** *of problem* (1.1) *if $u \in H_{loc}^{1,a}(\mathbb{R}^N) \cap L_{loc}^\infty(\mathbb{R}^N)$ and*

$$\int_{\mathbb{R}^N} Du \cdot D\varphi |x|^a \, dx = \int_{\mathbb{R}^N} \varphi |x|^b u^p \, dx \ \forall \text{non-negative } \varphi \in C_c^\infty(\mathbb{R}^N). \quad (1.3)$$

Owing to elementary elliptic regularity theory, weak solutions are at least of the class $C^2(\mathbb{R}^N \backslash \{0\})$ [9, 11]. Due to this, and unless we specify otherwise, we assume solutions are suitably regular in the following sense.

Definition. *We say a positive function u is a* **regular solution** *of problem* (1.1) *if u belongs to $C^2(\mathbb{R}^N\backslash\{0\}) \cap C(\mathbb{R}^N)$ and satisfies the equation pointwise everywhere in $\mathbb{R}^N\backslash\{0\}$. Furthermore, we say a regular solution u of* (1.1) *is* **stable** *if*

$$\int_{\mathbb{R}^N} |x|^a |D\varphi|^2 \, dx \geq p \int_{\mathbb{R}^N} |x|^b |u|^{p-1} \varphi^2 \, dx \text{ for all } \varphi \in C_c^1(\mathbb{R}^N). \quad (1.4)$$

As with the model equations, several distinct critical exponents arise when studying the existence of the different notions of solution to problem (1.1). Not surprisingly, we shall see that these critical exponents also appear when we characterize the quantitative and qualitative properties of the radial solutions. The first set of exponents are the familiar Serrin type exponent $p_{se}(a,b)$ and Sobolev type critical exponent $p_S(a,b)$, which are defined by

$$p_{se}(a,b) = \begin{cases} +\infty, & \text{if } N-2+a=0, \\ \dfrac{N+b}{N-2+a}, & \text{if } N-2+a>0, \end{cases} \quad (1.5)$$

and

$$p_S(a,b) = \begin{cases} +\infty, & \text{if } N-2+a=0, \\ \dfrac{N+2+2b-a}{N-2+a}, & \text{if } N-2+a>0. \end{cases} \quad (1.6)$$

We define a third exponent, a Joseph-Lundgren type critical exponent $p_{JL}(a,b)$, by

$$p_{JL}(a,b) = \begin{cases} +\infty, & \text{if } N \le 10+4b-5a, \\ P_+, & \text{if } N > 10+4b-5a, \end{cases} \tag{1.7}$$

where

$$P_+ = \tfrac{(N-2+a)^2-2(2+b-a)(N+b)+2\sqrt{(2+b-a)^3(2(N+b)-(2+b-a))}}{(N-2+a)(N-10-4b+5a)}.$$

Thus, for $N \ge 2$ and $b > a-2 > -N$, there holds

$$1 < p_{se}(a,b) < p_S(a,b) < p_{JL}(a,b) \le +\infty.$$

Under suitable assumptions, $p_{JL}(a,b)$ is the largest positive zero of the map

$$f(p) = p\Big(\frac{2+b-a}{p-1}\Big)\Big(N-2+a-\frac{2+b-a}{p-1}\Big)-\Big(\frac{N-2+a}{2}\Big)^2. \tag{1.8}$$

Remark 1. *The approach (and further details) in deriving $p_{JL}(a,b)$ from* (1.8) *may be found in [9, 11, 12]. For $a,b=0$, this critical exponent first appeared in [24], and its connection to finite Morse index and stable solutions along with other related results can be found in [12] (also see [11, 20]). Alternatively, $p_{JL}(a,b)$ may be derived from a stability analysis of an ODE system related to* (1.1) *(for details, see* (2.19) *in the proof of Lemma 3 below).*

The terminology adopted for the three critical exponents is inspired by the similar terms coined for the Lane-Emden equation ($a,b=0$) and the Hardy-Hénon equation ($a=0$). Not surprisingly, just as with those two model cases, the Serrin type, the Sobolev type and the Joseph-Lundgren type exponent are the dividing numbers for the sharp existence of singular, regular, and stable solutions for problem (1.1), respectively. This collection of sharp existence results is already well-established, but we summarize them in the following theorem; meanwhile, for more details on the special cases and other closely related results, the interested reader is referred to [16, 29, 31] for results pertaining to the Serrin exponent, [3, 4, 7, 14, 17, 18] for the Sobolev exponent, and [9, 12, 20, 24] for the Joseph-Lundgren exponent.

Theorem B. *Let $p>1$, $N \ge 2$ and suppose that $b > a-2 > -N$. Then*

(a) Problem (1.1) *admits a C^2 classical solution in $\mathbb{R}^N\backslash\{0\}$ if and only if $p > p_{se}(a,b)$.*

(b) Problem (1.1) *admits a regular solution if and only if $p \ge p_S(a,b)$.*

(c) *Problem* (1.1) *admits a non-trivial (non-negative or not) stable solution if and only if* $p \geq p_{JL}(a,b)$.

Part (a) of Theorem B is a special case of Theorem 4.1 in [21] and we point out the result extends to unbounded solutions. The proof of part (b) may be found in [22] and part (c) was established in [11]. We may extend the assertion in part (b) to the borderline case $N-2+a=0$. That is, since $p_S(a,b)=+\infty$ if $N-2+a=0$, we show no regular solutions exist for each $1<p<\infty$ when $N-2+a=0$ (see [11] and Proposition 1 below, although a slightly more general result is provided). It should be understood that part (c) in Theorem B is asserting that no non-trivial stable solutions exist for all $1<p<\infty$ for spatial dimension within the range $0<N-2+a\leq 4(2+b-a)$. As a consequence of Theorem B and the above remarks, we assume hereafter that $N-2+a>0$. The following result partly justifies the assumptions we place on the weighted coefficients of (1.1). The result can be found in [11], but we provide a proof for completeness sake.

Theorem 1. *Let* $N \geq 2$, $p>1$, $b>-N$ *and* $a-2>-N$. *Then* (1.1) *does not admit any weak solution provided that* $b \leq a-2$.

This result indicates that a necessary condition for the existence of weak solutions to (1.1) is $b>a-2>-N$, which we assume hereafter unless specified otherwise.

Remark 2. *Let us assume* $p>p_{se}(a,b)$ *and thus* $N-2+a>\beta>0$, *where*

$$\beta := \frac{2+b-a}{p-1}. \tag{1.9}$$

The function $u(x)=U_s(r)$, *where* $r=|x|$ *and*

$$U_s(r)=\Big[\beta(N-2+a-\beta)\Big]^{\frac{1}{p-1}} r^{-\beta}, \tag{1.10}$$

defines an unbounded classical solution of (1.1) *in punctured space* $\mathbb{R}^N\backslash\{0\}$. *We shall see that this singular solution plays an important role in the asymptotic properties of the radially symmetric solutions.*

Our main goal in this paper is to supplement Theorem B with a comprehensive analysis of positive radial solutions of (1.1), but let us clarify our notion of a radial solution here. The following observation motivates our definition below. If we set

$$r=|x|>0 \ \text{ and } \ \omega = x/|x| \in \mathbb{S}^{N-1} := \{\omega \in \mathbb{R}^N : |\omega|=1\}, \tag{1.11}$$

and write $u(x) = v(r, \omega)$, elementary calculations will reveal that $v = v(r, \omega)$ is a smooth solution of

$$r^a \Big(\frac{\partial^2 v}{\partial r^2} + \Big(\frac{N-1+a}{r} \Big) \frac{\partial v}{\partial r} + \frac{1}{r^2} \Delta_\omega v \Big) + r^b v^p = 0, \; v > 0, \;\text{ in } (0, \infty) \times \mathbb{S}^{N-1}, \tag{1.12}$$

where Δ_ω is the Laplace-Beltrami operator on the unit sphere $\mathbb{S}^{N-1}$.

Definition. *In view of the above, when referring to a radial or a radially symmetric solution u of (1.1), we mean $u(|x|) = v_\alpha(r)$ where α is some positive real and*

$$v_\alpha \in C^2((0, \infty)) \cap C([0, \infty))$$

satisfies the initial value problem

$$\begin{cases} \dfrac{d^2 v_\alpha}{dr^2} + \Big(\dfrac{N-1+a}{r} \Big) \dfrac{d v_\alpha}{dr} + r^{b-a} v_\alpha^p = 0, \; v_\alpha(r) > 0, \; r > 0, \\ v_\alpha(0) = \alpha > 0. \end{cases}$$

In our remaining two theorems, Theorem B dictates we take $p \geq p_S(a, b)$. Now our second theorem indicates that all radial solutions can be classified in the critical case $p = p_S(a, b)$. At the same time, it also shows these solutions decay near infinity with the fast rate $N - 2 + a$, while radial solutions in the supercritical setting decay with the slow rate β (as $N - 2 + a > \beta$).

Theorem 2. *Let $N \geq 2$, $b > a - 2 > -N$, and suppose that $p \geq p_S(a, b)$. Then for each $\alpha > 0$, there exists a unique radially symmetric solution of (1.1) with $u(0) = \alpha$, and this solution satisfies*

$$u(x) = \alpha v_1(\alpha^{1/\beta} |x|).$$

Given a radially symmetric solution u of (1.1), if $p > p_S(a, b)$, then

$$\lim_{|x| \longrightarrow +\infty} |x|^\beta u(x) = \Big[\beta(N - 2 + a - \beta) \Big]^{\frac{1}{p-1}},$$

and u is stable provided that $p \geq p_{JL}(a, b)$.

If $p = p_S(a, b)$, then

$$v_1(r) = \Big(\frac{(N+b)(N-2+a)}{(N+b)(N-2+a) + r^{2+b-a}} \Big)^{\frac{N-2+a}{2+b-a}} \tag{1.13}$$

and thus

$$\lim_{|x| \longrightarrow +\infty} |x|^{N-2+a} u(x) = \frac{1}{\alpha} \Big((N+b)(N-2+a) \Big)^{\frac{N-2+a}{2+b-a}}.$$

More can be said regarding the behavior of radial solutions, namely, on their intersecting properties. The subsequent theorem may be regarded as a refined asymptotic analysis of radial solutions for (1.1). It confirms that any radial solution will intersect another or the singular solution U_s finitely-many times if $p = p_S(a,b)$, while it will intersect either solution infinitely-many times if $p_S(a,b) < p < p_{JL}(a,b)$. If $p \geq p_{JL}(a,b)$ no intersection occurs; therefore, the radially symmetric solutions of (1.1) are ordered according to their initial values. Here, we let u_α denote the unique radially symmetric solution of (1.1) with initial condition $u(0) = \alpha$. We define the intersection number or the zero number between two distinct solutions u_{α_1} and u_{α_2} in $(0, \infty)$ by

$$\mathcal{Z}_{(0,\infty)}(u_{\alpha_1} - u_{\alpha_2}) := \#\{r \in (0,\infty) \,|\, u_{\alpha_1}(r) = u_{\alpha_2}(r)\}.$$

Theorem 3. *Let $N \geq 2$, $b > a - 2 > -N$ and $p \geq p_S(a,b)$, and suppose $\alpha_1, \alpha_2 > 0$ is a pair of distinct reals.*

(a) If $p_S(a,b) < p < p_{JL}(a,b)$, then

$$\mathcal{Z}_{(0,\infty)}(u_{\alpha_1} - u_{\alpha_2}) = \infty \text{ and } \mathcal{Z}_{(0,\infty)}(u_{\alpha_1} - U_s) = \infty.$$

(b) If $p \geq p_{JL}(a,b)$, then $\mathcal{Z}_{(0,\infty)}(u_{\alpha_1} - u_{\alpha_2}) = 0$, i.e., $\alpha_1 < \alpha_2$ implies

$$u_{\alpha_1}(r) < u_{\alpha_2}(r) \quad \text{for all} \quad r \geq 0.$$

(c) If $p = p_S(a,b)$, then

$$\mathcal{Z}_{(0,\infty)}(u_{\alpha_1} - u_{\alpha_2}) = 1 \text{ and } \mathcal{Z}_{(0,\infty)}(u_{\alpha_1} - U_s) = 2.$$

We emphasize that the existence and classification results in the case $p = p_S(a,b)$ were obtained earlier in [6]. The classification of finite-energy entire solutions of equation (1.1) in the 'symmetry region' (which includes the optimizers) was later established in [10]. This resolved a longstanding conjecture regarding the optimal symmetry range for optimizers and, in particular, complements the symmetry-breaking observed in [6] and [13]. We also point out that the existence and slow decay property of radial solutions in the supercritical range $p > p_S(a,b)$ given in Theorem 2 were obtained in [11]; however, we provide the complete proofs here for completeness and because our approach gives a unified approach for obtaining both sets of theorems.

The remainder of this paper is organized into two main sections. The first, Section 2, contains the proof of Theorem 1 then establishes several preliminary results concerning the local existence and local asymptotics of radially symmetric solutions. The section further provides a key phase plane stability analysis of a closely related first-order ODE system which we invoke to establish the refined asymptotic study of radial solutions to (1.1). The proofs of Theorem 2 and Theorem 3 are then given in the remaining section, Section 3.

2 Preparations and the Proof of Theorem 1

2.1 Proof of Theorem 1 and Proposition 1

Proof of Theorem 1 . If u is a positive weak solution of (1.1), then elliptic regularity theory ensures $u \in C^\infty(\mathbb{R}^N \setminus \{0\})$ and u satisfies (1.1) pointwise everywhere in any punctured ball domain $B_R(0)\setminus\{0\}$. Writing $u = u(r,\theta)$ in polar coordinates so that $u(r,\theta)$ satisfies (1.12), we then set

$$U(r) = \fint_{\mathbb{S}^{N-1}} u(r,\theta)\, dS := \frac{1}{|\mathbb{S}^{N-1}|} \int_{\mathbb{S}^{N-1}} u(r,\theta)\, dS \quad \text{for } 0 < r < R.$$

By Jensen's inequality,

$$\fint_{\mathbb{S}^{N-1}} u(r,\theta)^p\, dS \geq \Big(\fint_{\mathbb{S}^{N-1}} u(r,\theta)\, dS \Big)^p = U(r)^p. \tag{2.1}$$

Then (2.1) implies that $U(r) > 0$ satisfies the differential inequality

$$-\Big(U''(r) + \frac{N-1+a}{r} U'(r) \Big) \geq r^{b-a} U(r)^p \quad \text{for } 0 < r < R. \tag{2.2}$$

This implies that, for $0 < r < R$,

$$-\left(r^{N-1+a} U'(r)\right)' \geq r^{N-1+b} U(r)^p. \tag{2.3}$$

Since $U > 0$, we obtain $-(r^{N-1+a}U'(r))' > 0$ and thus

$$r^{N-1+a} U'(r) \longrightarrow \ell \quad \text{as } r \longrightarrow 0^+,$$

where $-\infty < \ell \leq \infty$. We infer that $\ell \leq 0$. Otherwise, if $0 < \ell \leq +\infty$, then we can find $\delta > 0$ and $r_\delta > 0$ such that

$$U'(r) \geq \delta r^{-(N+a-1)} \quad \text{for } 0 < r < r_\delta.$$

Integrating this in (r_0, r) where $0 < r_0 < r < r_\delta$ yields

$$U(r) \geq \delta \int_{r_0}^{r} t^{-(N+a-2)}\, \frac{dt}{t}.$$

As $N-2+a>0$, sending $r_0 \longrightarrow 0^+$ leads to an impossibility.

Let $\ell \le 0$. Therefore, we have that $U'(r) < 0$. Indeed, there exists a positive constant c and $r_1 > 0$ such that

$$U(r) \ge c \text{ for } 0 < r < r_1. \tag{2.4}$$

Choose a small $r_0 \in (0, r_1)$. By integrating (2.3) in $(r_0, r) \subset (r_0, r_1)$ and since U is monotone decreasing in this interval of integration, we obtain

$$\begin{aligned} -r^{N-1+a}U'(r) &\ge r_0^{N-1+a}U'(r_0) - r^{N-1+a}U'(r) \\ &\ge U(r)^p \int_{r_0}^{r} t^{N+b}\,\frac{dt}{t} \text{ for } 0 < r_0 < r < r_1. \end{aligned} \tag{2.5}$$

Note that if $N+b \le 0$, then sending $r_0 \longrightarrow 0^+$ leads to a contradiction, since the integral on the right diverges. Now, as $N+b>0$, we may integrate then send $r_0 \longrightarrow 0^+$ in (2.5) to get

$$-U'(r)U(r)^{-p} \ge (N+b)^{-1} r^{1+b-a} \quad \text{for } 0 < r < r_1.$$

That is, for each $\epsilon_0 > 0$,

$$(p-1)^{-1}\Big(U(r)^{-(p-1)}\Big)' \ge (N+b)^{-1} r^{1+b-a} \quad \text{for } 0 < \epsilon_0 < r < r_1.$$

Integrating once again in the interval (ϵ_0, r) yields

$$U(r)^{-(p-1)} \ge \frac{p-1}{N+b} \int_{\epsilon_0}^{r} t^{2+b-a}\,\frac{dt}{t}.$$

Now, after sending $\epsilon_0 \longrightarrow 0^+$ in the last estimate, the resulting integral diverges if $b \le a-2$. This is impossible due to (2.4), and this completes the proof of the Theorem. □

Proposition 1. *Let $p>1$, $N \ge 2$ and $b > a-2$. If $N-2+a>0$ and $p \le p_{se}(a,b)$, then (1.1) admits no regular solution. If $N-2+a=0$, then the same conclusion holds for each $1<p<\infty$.*

Proof. We prove this by contradiction. We assume u is a regular solution of (1.1).
Step 1: Integral Estimates.

Fix $R > 0$ and choose a test function $\xi \in C_c^\infty(B_2(0))$ such that $0 \le \xi \le 1$ and $\xi \equiv 1$ in $B_1(0)$. Take $\varphi(x) = \xi(x/R)^{2p^*}$, where $p^* = p/(p-1)$. It is simple to check there exists a constant $C > 0$ such that

$$
\begin{aligned}
|div(|x|^a D\varphi)| &= |x|^a \Big|\Delta\varphi + a|x|^{-2}(x \cdot D\varphi)\Big| \\
&\le \frac{C|x|^a}{R^2}\xi(\frac{x}{R})^{2(p^*-1)}\Big(\xi(\frac{x}{R})|\Delta\xi(\frac{x}{R})| + |D\xi(\frac{x}{R})|^2 + |D\xi(\frac{x}{R})|\Big) \\
&\le CR^{-2}|x|^a\varphi(x)^{1/p} \quad \text{for} \quad R \le |x| < 2R.
\end{aligned}
$$

Multiplying the equation in (1.1) by φ then integrating over $\mathbb{R}^N$, integration by parts and Hölder's inequality imply

$$
\begin{aligned}
\int_{\mathbb{R}^N} \varphi|x|^b u^p\,dx = \int_{\mathbb{R}^N} u(-div(|x|^a D\varphi)\,dx &\le \frac{C}{R^2}\int_{\Omega_R} \varphi^{1/p}|x|^a u\,dx \\
&\le \frac{C}{R^2}\Big(\int_{\Omega_R} |x|^{(a-b/p)p^*}\,dx\Big)^{1/p^*}\Big(\int_{\Omega_R} \varphi|x|^b u^p\,dx\Big)^{1/p} \\
&\le CR^{-2+\frac{N}{p^*}+a-\frac{b}{p}}\Big(\int_{\Omega_R} \varphi|x|^b u^p\,dx\Big)^{1/p},
\end{aligned}
$$

where $1/p + 1/p^* = 1$ and $\Omega_R := B_{2R}(0)\backslash B_R(0)$. It follows that there exists a constant $C > 0$ depending on N, p, a and b such that

$$
\int_{B_R(0)} |x|^b u^p\,dx \le CR^{-2+\frac{N}{p^*}+a-\frac{b}{p}}\Big(\int_{B_{2R}(0)\backslash B_R(0)} |x|^b u^p\,dx\Big)^{1/p}, \tag{2.6}
$$

and

$$
\int_{B_R(0)} |x|^b u^p\,dx \le CR^{N+p^*(a-\frac{b}{p}-2)}. \tag{2.7}
$$

Step 2: If $N-2+a \ge 0$ and $p < p_{se}(a,b) \le +\infty$, then $N+p^*(a-\frac{b}{p}-2) < 0$. Sending $R \longrightarrow \infty$ in (2.7) will show $u \equiv 0$. If $N-2+a > 0$ and $p = p_{se}(a,b)$, then the same reasoning yields $u \in L_b^p(\mathbb{R}^N)$. Then, sending $R \longrightarrow \infty$ in (2.6) allows us to conclude that $\|u\|_{L_b^p(\mathbb{R}^N)} = 0$. In either case, we arrived at a contradiction, and this completes the proof.

$\square$

2.2 Local existence of radially symmetric solutions

Recall to find radially symmetric solutions of (1.1), we consider the second-order equation

$$v''(r) + \Big(\frac{N-1+a}{r}\Big)v'(r) + r^{b-a}v(r)^p = 0,\ v(r) > 0,\ \text{ in } (0, r_*) \subset \mathbb{R}. \quad (2.8)$$

Therefore, given $\alpha > 0$, our goal is to eventually find a unique global solution v of (2.8) with $v(0) = \alpha$ and $r_* = +\infty$. Moreover, the following Emden-Fowler type transformation will be useful. If we set

$$t = \log r \ \text{ and } \ w(t) = r^\beta v(r), \quad (2.9)$$

then w satisfies

$$w'' + \Lambda_1 w' - \Lambda_2 w + w^p = 0,\ w > 0,\ \text{ for } t \in (-\infty, \log r_*), \quad (2.10)$$

where

$$\Lambda_1 = N - 2 + a - 2\beta \ \text{ and } \ \Lambda_2 = \beta(N - 2 + a - \beta).$$

Here, the notation $'$ will either denote d/dr or d/dt when the context is clear, and we note that $p \geq p_S(a, b)$ implies that $\Lambda_1 \geq 0$ and $\Lambda_2 > 0$. Another key component of our proofs of the main results will rely on the following energy functional

$$E(w) := E(w, w') = \frac{1}{2}(w')^2 - \frac{\Lambda_2}{2}w^2 + \frac{1}{p+1}w^{p+1}, \quad (2.11)$$

which satisfies $E(0, 0) = 0$ and

$$\frac{d}{dt}E(w(t)) = -\Lambda_1(w'(t))^2 \leq 0 \text{ for all } t. \quad (2.12)$$

Lemma 1. *Let $N \geq 2$, $b > a - 2 > -N$ and suppose that $p \geq p_S(a, b)$. For each $\alpha > 0$, the initial-value problem* (2.8) *with initial condition $v(0) = \alpha$ admits a unique local solution.*

Proof. The existence of a local solution will follow from a standard fixed point argument, but we shall first require the following.

Step 1: For each local solution v of (2.8) with $v(0) = \alpha$, we necessarily have that

$$\lim_{r \to 0^+} r^{N-1+a}v'(r) = 0. \quad (2.13)$$

The proof of this claim has many points in common with the proof of Theorem 1. First observe that from (2.8), v must satisfy

$$-\left(r^{N-1+a}v'(r)\right)' = r^{N-1+b}v(r)^p > 0 \ \text{ for } r > 0. \quad (2.14)$$

Thus, $r^{N-1+a}v'(r)$ is monotone decreasing and therefore

$$\lim_{r\to 0^+} r^{N-1+a}v'(r) = \ell \in (-\infty,\infty]. \tag{2.15}$$

We assert that $\ell = 0$. Otherwise, if $\ell \in (0,\infty]$, then we can find $0 < c < \ell$ and $r_0 > 0$ such that

$$v'(r) \geq cr^{-(N-1+a)} \text{ for } 0 < r < r_0.$$

By choosing $0 < \epsilon_0 < r < r_0$, integrating the above inequality over (ϵ_0, r) leads to

$$v(r) \geq v(r) - v(\epsilon_0) \geq \frac{c}{(N-2+a)}\Big(\epsilon_0^{-(N-2+a)} - r_0^{-(N-2+a)}\Big).$$

Sending $\epsilon_0 \longrightarrow 0^+$ above will lead to an obvious contradiction. Now let us assume $\ell \in (-\infty, 0)$. In this case, we can find $r_1 > 0$ such that

$$r^{N-1+a}v'(r) < \ell/2 < 0 \text{ for } 0 < r < r_1.$$

Integrating this over (r, r_1) and rearranging terms gives us

$$v(r) \geq v(r_1) - \frac{\ell}{2(N-2+a)}\Big(r^{-(N-2+a)} - r_1^{-(N-2+a)}\Big) \text{ for } 0 < r < r_1.$$

Hence, for all suitably small $0 < r < r_1$,

$$v(r) \geq C_1 r^{-(N-2+a)}, \tag{2.16}$$

where C_1 is some positive constant. On the other hand, if we choose $0 < \epsilon_1 < r < r_1$, noting that $\epsilon_1^{N-1+a}v'(\epsilon_1) \leq 0$, and integrating (2.14) over the interval (ϵ_1, r), we get

$$-r^{N-1+a}v'(r) \geq \epsilon_1^{N-1+a}v'(\epsilon_1) - r^{N-1+a}v'(r) = \int_{\epsilon_1}^{r} s^{N-1+b}v(s)^p\, ds.$$

From this, after sending $\epsilon_1 \longrightarrow 0$, we obtain

$$-r^{N-1+a}v'(r) \geq \frac{r^{N+b}}{N+b}v(r)^p \text{ for } 0 < r < r_1;$$

which further implies there is a constant $C_2 > 0$ such that

$$v(r)^{-(p-1)} \geq C_2 r^{2+b-a} \text{ for } 0 < r < r_1.$$

Thus,

$$v(r) \le C_3 r^{-\beta} \ \text{ for } 0 < r < r_1.$$

Combining this with (2.16) proves there is a constant $C > 0$ such that

$$r^{N-2+a-\beta} \ge C \ \text{ for all suitably small } r > 0.$$

But this cannot happen since $N - 2 + a > \beta$ due to $p \ge p_S(a,b) > p_{se}(a,b)$. Hence, $\ell = 0$ and (2.13) follows.

Step 2: Consider the integral operator $T : C([0, r_*]) \longrightarrow C([0, r_*])$ where

$$Tv(r) = \alpha - \int_0^r \int_0^t \Big(\frac{s^{N-1+b}}{t^{N-1+a}}\Big) v(s)^p \, ds \, dt.$$

From (2.8), we see that $-(r^{N-1+a} v'(r))' = r^{N-1+b} v(r)^p$. By choosing $0 < \epsilon < r$ and integrating the previous equation in the interval (ϵ, r), we obtain

$$\epsilon^{N-1+a} v'(\epsilon) - r^{N-1+a} v'(r) = \int_\epsilon^r s^{N-1+b} v(s)^p \, ds. \tag{2.17}$$

In view of (2.13), sending $\epsilon \longrightarrow 0^+$ leads to

$$-r^{N-1+a} v'(r) = \int_0^r s^{N-1+b} v(s)^p \, ds \ \text{ for } r > 0.$$

From this, we see that $v(r)$ is a local solution of (2.8) with $v(0) = \alpha$ if and only if it is a fixed point of T. Indeed, the existence of a unique positive fixed point of T follows from the contraction mapping principle provided $r_* > 0$ is chosen sufficiently small.

□

Lemma 2. *For each $\alpha > 0$, let $v = v(r)$ be the unique local solution of* (2.8) *with $v(0) = \alpha$. If $w(t)$ is defined by* (2.9)*, then (w, w') converges to $(0,0)$ as $t \longrightarrow -\infty$.*

Proof. Clearly, w tends to 0 as $t \longrightarrow -\infty$. By rewriting the equation in (2.8) into

$$-(r^{N-1+a} v'(r))' = r^{N-1+b} v(r)^p$$

and keeping (2.13) in mind, we obtain

$$-r^{N-1+a} v'(r) = \int_0^r s^{N-1+b} v(s)^p \, ds > 0.$$

Thus

$$0 < -rv'(r) = \frac{1}{r^{N-2+a}} \int_0^r s^{N-1+b} v(s)^p \, ds \le Cr^{2+b-a} \quad \text{for small } r > 0.$$

In view of $w'(t) = r^\beta(\beta v(r) + rv'(r))$, we conclude that

$$|w'(t)| \le C(w(t) + r^{\beta+2+b-a})$$

where $\beta + 2 + b - a = p\beta > 0$ and thus $w'(t) \longrightarrow 0$ as $t \longrightarrow -\infty$. $\square$

2.3 Stability analysis of radially symmetric solutions

By taking $x_1 = w$ and $x_2 = w'$, we may convert equation (2.10) into the two-dimensional nonlinear system

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = F(x_1, x_2) := \begin{pmatrix} 0 & 1 \\ \Lambda_2 & -\Lambda_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 0 \\ x_1^p \end{pmatrix}. \tag{2.18}$$

The equilibria of (2.18) are

$$(x_1, x_2) = \mathcal{O} := (0,0) \text{ and } (x_1, x_2) = \mathcal{S} := (\Lambda_2^{\frac{1}{p-1}}, 0).$$

To study their stability, we note that

$$A := DF(0,0) = \begin{pmatrix} 0 & 1 \\ \Lambda_2 & -\Lambda_1 \end{pmatrix} \text{ and } B := DF(\Lambda_2^{\frac{1}{p-1}}, 0) = \begin{pmatrix} 0 & 1 \\ -\Lambda_2(p-1) & -\Lambda_1 \end{pmatrix}.$$

The eigenvalues and corresponding eigenvectors of A are

$$\lambda_\pm = -\frac{\Lambda_1}{2} \pm \frac{\sqrt{\Lambda_1^2 + 4\Lambda_2}}{2}, \; v_\pm = \begin{pmatrix} \lambda_\mp \\ -\Lambda_2 \end{pmatrix},$$

and for B, they are

$$\lambda_\pm = -\frac{\Lambda_1}{2} \pm \frac{\sqrt{\Lambda_1^2 - 4\Lambda_2(p-1)}}{2}, \; v_\pm = \begin{pmatrix} \lambda_\mp \\ \Lambda_2(p-1) \end{pmatrix}.$$

Lemma 2 indicates we should examine system (2.18) subject to the conditions

$$x_1 > 0 \text{ and } (x_1, x_2) \longrightarrow (0,0) \text{ as } t \longrightarrow -\infty.$$

Now, given a solution (x_1, x_2) of (2.18) we consider its orbit, which we denote by Γ, that emanates from the origin $(0,0)$ in the right half-space $x_1 > 0$ of the x_1x_2-plane. This trajectory Γ represents the graph of a solution of (2.10) on the ww'-plane. Due to $E(0,0) = 0$, $E(0,y) > 0$ for all $y \neq 0$, and from the monotonicity of the energy functional indicated by (2.12), we first conclude that Γ cannot hit the x_2-axis and so this orbit must remain in the right half-space; and secondly, the corresponding solution $w(t)$ must exist globally in $\mathbb{R}$ and both $w(t)$ and $w'(t)$ remain bounded for all $t \in \mathbb{R}$. It follows that w' and w'' vanish as $t \longrightarrow \infty$ and therefore Γ must converge to either $(0,0)$ or $(\Lambda_2^{1/(p-1)}, 0)$ (for more details, the reader is referred to, e.g., Lemmas 3.1-3.2 in [25]).

We now establish a phase plane analysis for system (2.18), and this will comprise an important ingredient in our proofs of Theorem 2 and Theorem 3.

Lemma 3. *Let $N \geq 2$, $b > a-2 > -N$ and suppose that $p \geq p_S(a,b)$.*

(a) If $p > p_S(a,b)$, then $\mathcal{O} = (0,0)$ is a saddle and $\mathcal{S}$ is stable. Furthermore,

(i) if $p_S(a,b) < p < p_{JL}(a,b)$, then $\mathcal{S} = (\Lambda_2^{1/(p-1)}, 0)$ is a stable focus and Γ spirals towards $\mathcal{S}$.

(ii) If $p \geq p_{JL}(a,b)$, then Γ is a heteroclinic orbit in the first quadrant $\{x_2 > 0\}$ from $\mathcal{O}$ to $\mathcal{S}$, and x_1 is monotone increasing along the orbit Γ as t increases from $-\infty$ to $+\infty$.

(b) If $p = p_S(a,b)$, then Γ is a homoclinic orbit of the stable equilibrium $\mathcal{O}$.

Proof. **Case (a) $p > p_S(a,b)$.**

The stability of the equilibria just follows from the fact that the eigenvalues of A are real with $\lambda_- < 0 < \lambda_+$ and the eigenvalues of B have negative real parts. To get a better picture of the flow generated by system (2.18), observe the x_1-axis is the vertical nullcline and since $\Lambda_1 > 0$ in this case, the horizontal nullcline is given by the curve $x_2 = -\Lambda_1^{-1} x_1 (x_1^{p-1} - \Lambda_2)$ that lies on right half-plane and passes through both equilibria.

As $\mathcal{O}$ is a saddle point, while noting the direction of the corresponding eigenvectors of A, and because the flow along nullclines changes precisely across the line $x_1 = \Lambda_2^{1/(p-1)}$, we see that the flow generated by the system ensures Γ emanates and increases away from the origin initially in the first quadrant. Although we know Γ converges to $\mathcal{S}$, we shall confirm only two scenarios arise. Namely, either Γ spirals towards $\mathcal{S}$ if $p < p_{JL}(a,b)$, or else x_1 increases along Γ, which follows the horizontal nullcline as it converges to $\mathcal{S}$.

Subcase (i) $p_S(a,b) < p < p_{JL}(a,b)$. Indeed, the eigenvalues of B are complex conjugates with negative real parts. Hence, $\mathcal{S}$ is a stable spiral.

Subcase (ii) $p \geq p_{JL}(a,b)$. We defined the Joseph-Lundgren type exponent $p_{JL}(a,b)$ to ensure that $p \geq p_{JL}(a,b)$ implies that the eigenvalues of B are real, since

$$\Lambda_1^2 \geq 4\Lambda_2(p-1). \tag{2.19}$$

Since $\mathcal{O}$ is a saddle point and noting the direction of its unstable eigenvector, Γ increases away from the origin in the first quadrant initially. Now along the line

$$x_2 = -\frac{\Lambda_1}{2}(x_1 - \Lambda_2^{\frac{1}{p-1}}) \text{ for } x_1 < \Lambda_2^{\frac{1}{p-1}} \tag{2.20}$$

that passes through the equilibrium point $\mathcal{S}$, we show Γ must remain in the first quadrant but then decreases towards $\mathcal{S}$. Indeed, Γ always lies above the x_1-axis due to the increasing direction along this vertical nullcline for $x_1 < \Lambda_2^{1/(p-1)}$, and on the line (2.20) the slope of the trajectory satisfies

$$\begin{aligned}
\frac{x_2'}{x_1'} &= -\Lambda_1 - \frac{x_1}{x_2}\Big(x_1^{p-1} - \Lambda_2\Big) = -\Lambda_1 + \frac{2x_1(x_1^{p-1} - \Lambda_2)}{\Lambda_1(x_1 - \Lambda_2^{\frac{1}{p-1}})} \\
&= -\Lambda_1 + \frac{2}{\Lambda_1}x_1(p-1)\xi^{p-2} \text{ (for some } \ x_1 < \xi < \Lambda_2^{\frac{1}{p-1}}) \\
&< -\Lambda_1 + \frac{2}{\Lambda_1}(p-1)\Lambda_2 \leq -\frac{\Lambda_1}{2},
\end{aligned}$$

where we used the mean value theorem and the fact that $\frac{2}{\Lambda_1}(p-1)\Lambda_2 \leq \frac{\Lambda_1}{2}$ thanks to (2.19). This guarantees the incoming trajectory remains below the line (2.20). Thus, we deduce that Γ remains in the first quadrant and, as $t \longrightarrow +\infty$, $x_1(t)$ increases along Γ as it tends to $\Lambda_2^{1/(p-1)}$.

Case (b) $p = p_S(a,b)$**.** In this case, the eigenvalues of B are purely complex conjugates. Since $\Lambda_1 = 0$ in this case, there holds $E'(w(t)) = 0$ so that E is constant along any solution; that is, the orbit is given by the level curve

$$E(x_1, x_2) \equiv 0 = E(0,0). \tag{2.21}$$

Since $E(0, x_2) = x_2^2/2 > 0$ for all $x_2 \neq 0$ and

$$E(\Lambda_2^{\frac{1}{p-1}}, 0) = (\frac{1}{1+p} - \frac{1}{2})\Lambda_2^{\frac{p+1}{p-1}} < 0,$$

the orbit Γ lies on the curve (2.21), tending away $\mathcal{O}$ and traversing about $\mathcal{S}$ clockwise as it must converge back towards $\mathcal{O}$ as $t \longrightarrow \infty$. Hence, Γ is a homoclinic orbit.

□

3 Proof of Theorem 2 and Theorem 3

We are now in the position to prove the remaining theorems.

3.1 Proof of Theorem 2

Suppose that $N \geq 2$, $b > a - 2 > -N$ and $p \geq p_S(a,b)$.

Step 1 (Existence). Given $\alpha > 0$, the initial value problem,

$$\begin{cases} \dfrac{d^2v}{dr^2} + \Big(\dfrac{N-1+a}{r}\Big)\dfrac{dv}{dr} + r^{b-a}v^p = 0, \ v(r) > 0, \ r > 0, \\ v(0) = \alpha > 0, \end{cases} \tag{3.1}$$

admits a unique local solution of class $C^2((0,r_*)) \cap C([0,r_*))$ by Lemma 1. We denote this unique solution by v_α. By taking the Emden-Fowler type transformation $w_\alpha(t) = r^\beta v_\alpha(r)$ and recalling the discussion in the previous section, we can extend w_α globally and therefore we may extend v_α to a unique solution with $r_* = +\infty$.

Step 2 (Uniqueness). Let $u = u(|x|)$ be a radially symmetric solution of problem (1.1). Indeed, u satisfies (3.1) with $\alpha = u(0)$ and therefore $v_\alpha(r) = u(|x|)$. Observe that (1.1) is invariant under the scaling

$$u(x) \longrightarrow u_\lambda(x) := \lambda u(\lambda^{1/\beta}x),$$

i.e., for each $\lambda > 0$, u_λ remains a solution provided that u is a solution of (1.1). Consequently, $u(0)v_1(u(0)^{1/\beta}r)$ satisfies the same initial value problem as v_α, and hence, ODE uniqueness theory ensures that

$$v_\alpha(r) = \alpha v_1(\alpha^{1/\beta}r). \tag{3.2}$$

In particular, if $p = p_S(a,b)$ and $\alpha = 1$, then a routine calculation reveals $v_1(r)$ defined by (1.13) is the unique solution of (3.1) with initial value $\alpha = 1$.

Step 3. If $p > p_S(a,b)$, then Lemma 3 shows that any trajectory, including the solution curve (w,w') where $w(t) = r^\beta v_\alpha(r)$ and $\alpha = u(0)$, must converge to the equilibrium solution $(\Lambda_2^{1/(p-1)}, 0)$. Therefore,

$$\lim_{|x| \longrightarrow +\infty} |x|^\beta u(|x|) = \Lambda_2^{\frac{1}{p-1}}.$$

Now assume $p \geq p_{JL}(a,b)$ and let u be a radially symmetric solution of (1.1). Thanks to part (a.ii) of Lemma 3, we have that

$$u(|x|) \leq U_s(r) \text{ for all } r > 0$$

Hence, for each $\varphi \in C_c^1(\mathbb{R}^N)$,

$$\begin{aligned}
\int_{\mathbb{R}^N} |x|^a |D\varphi|^2\,dx - p \int_{\mathbb{R}^N} |x|^b u^{p-1} \varphi^2\,dx& \\
\geq \int_{\mathbb{R}^N} |x|^a |D\varphi|^2\,dx - p \int_{\mathbb{R}^N} |x|^b U_s^{p-1} \varphi^2\,dx& \\
= \int_{\mathbb{R}^N} |x|^a |D\varphi|^2\,dx - pC_{a,b,N,p} \int_{\mathbb{R}^N} |x|^{-2+a} \varphi^2\,dx&,
\end{aligned}$$

where

$$pC_{a,b,N,p} = p\beta(N-2+a-\beta) \leq \Big(\frac{N-2+a}{2}\Big)^2,$$

and this last inequality holds because $p \geq p_{JL}(a,b)$ implies $f(p) \leq 0$, where $f(p)$ was defined in (1.8). Then, from the endpoint case of the Caffarelli-Kohn-Nirenberg inequality (1.2), namely, when $N \geq 3$, $b = a-2$, and $p = 1$ there holds

$$\mathcal{N}(a,N) \int_{\mathbb{R}^N} |x|^{-2+a} \varphi^2\,dx \leq \int_{\mathbb{R}^N} |x|^a |D\varphi|^2\,dx \text{ for all } \varphi \in C_c^1(\mathbb{R}^N),$$

and the sharp constant is explicitly given by (see [6, 8] for details)

$$\mathcal{N}(a,N) = \Big(\frac{N-2+a}{2}\Big)^2.$$

Invoking this in the previous chain of estimates yields

$$\begin{aligned}
\int_{\mathbb{R}^N} |x|^a |D\varphi|^2\,dx - p \int_{\mathbb{R}^N} |x|^b u^{p-1} \varphi^2\,dx& \\
\geq \int_{\mathbb{R}^N} |x|^a |D\varphi|^2\,dx - pC_{a,b,N,p} \int_{\mathbb{R}^N} |x|^{-2+a} \varphi^2\,dx& \\
\geq \int_{\mathbb{R}^N} |x|^a |D\varphi|^2\,dx - \mathcal{N}(a,N) \int_{\mathbb{R}^N} |x|^{-2+a} \varphi^2\,dx \geq 0&.
\end{aligned}$$

This proves u is a stable solution of (1.1). This completes the proof of the theorem. $\square$

3.2 Proof of Theorem 3

We make the same assumptions and adopt the same notation used in Lemma 3. Given $\alpha > 0$, we denote the unique global solution of (3.1) by v_α and we set $w_\alpha(t) = r^\beta v_\alpha(r)$ where $t = \log r$. We choose any pair of positive reals $\alpha_1 < \alpha_2$.

Case (a) $p_S(a,b) < p < p_{JL}(a,b)$**.**

From the rescaling (3.2), the graphs of w_α and w_1 are identical and therefore lie on some orbit Γ. From part (a.i) of Lemma 3, this trajectory Γ emanates away from $(0,0)$ and spirals towards the stable focus $(\Lambda_2^{1/(p-1)}, 0)$. Specifically, by rescaling, we find a real number $\tau = \tau(\alpha)$ satisfying $\alpha = e^{-\beta\tau(\alpha)}$ so that

$$w_\alpha(t) = e^{\beta t}\alpha v_1(\alpha^{1/\beta}e^t) = e^{-\beta\tau+\beta t}v_1(e^{-\tau}e^t) = w_1(t-\tau) \text{ for all } t \in \mathbb{R}. \quad (3.3)$$

So we have that $v_{\alpha_1}(r) = v_{\alpha_2}(r)$ for some $r > 0$ if and only if

$$w_1(t-\tau(\alpha_1)) = w_1(t-\tau(\alpha_2)) \text{ for some } t \in \mathbb{R}. \quad (3.4)$$

Since Γ spirals towards the non-trivial equilibrium point $\mathcal{S}$, (3.4) of course happens for infinitely many points in t and thus the radial solutions intersect one another infinitely many times. Similar reasoning will show that $w_{\alpha_1}(t)$ crosses

$$W(t) = r^\beta U_s(r) \equiv \Lambda_2^{1/(p-1)}$$

infinitely many times. Hence, $v_{\alpha_1}(r)$ intersects the singular solution $U_s(r)$ at infinitely many points.

Case (b) $p \geq p_{JL}(a,b)$**.**

From (3.3) and part (a.ii) of Lemma 3, we see that $w_{\alpha_1}(t) < w_{\alpha_2}(t)$ for all $-\infty < t < \infty$. That is, $v_{\alpha_1}(r) < v_{\alpha_2}(r)$ for all $r > 0$.

Case (c) $p = p_S(a,b)$**.**

Imitating the earlier argument in part (a) and noting part (b) of Lemma 3, we can show there exists a unique time t such that $w_1(t-\tau(\alpha_1)) = w_1(t-\tau(\alpha_2))$. Hence, the corresponding solutions $v_{\alpha_1}(r)$ and $v_{\alpha_2}(r)$ intersect at only one point $\bar{r} > 0$. Alternatively, from the classification result of Theorem 2, we have that $v_{\alpha_1}(r) = v_{\alpha_2}(r)$ if and only if

$$\alpha_1 v_1(\alpha_1^{1/\beta} r) = \alpha_2 v_1(\alpha_2^{1/\beta} r).$$

This occurs at exactly one point in $(0,\infty)$; namely, when

$$\bar{r} = \Big[\frac{(N+b)(N+2-a)((\alpha_2/\alpha_1)^{(p-1)/2} - 1)}{\alpha_2^{p-1} - (\alpha_1\alpha_2)^{(p-1)/2}}\Big]^{\frac{1}{2+b-a}}.$$

Similar arguments will show that $w_{\alpha_1}(t) = \Lambda_2^{1/(p-1)}$ holds at two distinct times. This shows that each radially symmetric solution v_{α_1} twice interests the singular solution. This completes the proof of the theorem $\square$

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